Exercise 23

Given functions $p(x) = \frac{1}{\sqrt{x}}$ and $m(x) = x^2 - 4$, state the domain of each of the following functions using interval notation:

(a) $\frac{p(x)}{m(x)}$

(b) $p(m(x))$

(c) $m(p(x))$

Solution

Part (a)

Compute the function $p(x)/m(x)$.

$$\frac{p(x)}{m(x)} = \frac{\frac{1}{\sqrt{x}}}{x^2 - 4}$$
$$= \frac{1}{\sqrt{x}(x^2 - 4)}$$

It's impossible to divide by zero, and it's impossible to take the square root of a negative number.

$$x \geq 0 \quad \text{and} \quad \sqrt{x}(x^2 - 4) \neq 0$$

$$x \geq 0 \quad \text{and} \quad \left(\sqrt{x} \neq 0 \quad \text{or} \quad x^2 - 4 \neq 0\right)$$

$$x \geq 0 \quad \text{and} \quad \left[x \neq 0 \quad \text{or} \quad (x+2)(x-2) \neq 0\right]$$

$$x \geq 0 \quad \text{and} \quad \left(x \neq 0 \quad \text{or} \quad x + 2 \neq 0 \quad \text{or} \quad x - 2 \neq 0\right)$$

$$x \geq 0 \quad \text{and} \quad \left(x \neq 0 \quad \text{or} \quad x \neq -2 \quad \text{or} \quad x \neq 2\right).$$

Therefore, the domain of $p(x)/m(x)$ in interval notation is $(0, 2) \cup (2, \infty)$.

Part (b)

Compute $p(m(x))$ by plugging the formula for $m(x)$ where x is in the formula for $p(x)$.

$$p(m(x)) = \frac{1}{\sqrt{x^2 - 4}}$$

It's impossible to divide by zero, and it's impossible to take the square root of a negative number.

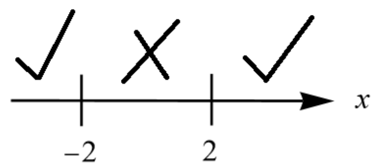
$$x^2 - 4 \geq 0 \quad \text{and} \quad \sqrt{x^2 - 4} \neq 0$$

$$(x+2)(x-2) \geq 0 \quad \text{and} \quad x^2 - 4 \neq 0$$

$$(x+2)(x-2) \geq 0 \quad \text{and} \quad x^2 \neq 4$$

$$(x+2)(x-2) \geq 0 \quad \text{and} \quad x \neq \pm 2$$

For the inequality on the left, the critical points are -2 and 2. Partition the number line at these numbers and test where the inequality is true.



Therefore, the domain of $p(m(x))$ in interval notation is $(-\infty, -2) \cup (2, \infty)$.

Part (c)

Compute $m(p(x))$ by plugging the formula for $p(x)$ where x is in the formula for $m(x)$.

$$m(p(x)) = \left(\frac{1}{\sqrt{x}}\right)^2 - 4 = \frac{1}{x} - 4$$

It's impossible to divide by zero, and it's impossible to take the square root of a negative number.

$$x \geq 0 \quad \text{and} \quad x \neq 0$$

Combine the two conditions.

$$x > 0$$

Therefore, the domain of $m(p(x))$ in interval notation is $(0, \infty)$.